

# EQUIVALENCE OF TWO DEFINITIONS OF SET-THEORETIC YANG-BAXTER HOMOLOGY

JÓZEF H. PRZYTICKI AND XIAO WANG

**ABSTRACT.** In 2004, Carter, Elhamdadi and Saito defined a homology theory for set-theoretic Yang-Baxter operators (we will call it the “algebraic” version in this article). In 2012, Przytycki defined another homology theory for pre-Yang-Baxter operators which has a nice graphic visualization (we will call it the “graphic” version in this article). We show that they are equivalent. The “graphic” homology is also defined for pre-Yang-Baxter operators, and we give some examples of its one-term and two-term homologies. In the two-term case, we have found torsion in homology of Yang-Baxter operator that yields the Jones polynomial.

## 1. INTRODUCTION

The Yang-Baxter equation was introduced independently by C.N. Yang(1967)[Yan] and R.J. Baxter(1972)[Bax]. It is well known that a certain solution of Yang-Baxter equation give rise to the Jones polynomial [Jon-2]. In 2004, Carter, Elhamdadi and Saito defined a (co)homology theory for set-theoretic Yang-Baxter operators, from which they gave a way to generate link invariants, cocycle invariants[CES]. In 2012, Przytycki gave a graphical definition of homology for a pre-Yang-Baxter operator [Prz-2]. We provide the definitions of two homology theories for set-theoretic Yang-Baxter operators in Section 2 and show their equivalence in Section 3. In Section 4, we give definitions of one-term and two-term homology of pre-Yang-Baxter operators, and show examples, in particular, we find torsion in two-term homology of Yang-Baxter operator that yields the Jones polynomial. We start from basic definitions.

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**Definition 1.1.** Let  $k$  be a commutative ring and  $X$  be a set. Consider  $V = kX$ , the free  $k$ -module generated by  $X$ . If a  $k$ -linear map,  $R : V \otimes V \rightarrow V \otimes V$ , satisfies the following equation  $(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V) = (Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R)$ , then we say  $R$  is a pre-Yang-Baxter operator. The equation above is called a Yang-Baxter equation. If, in addition,  $R$  is invertible, then we say  $R$  is a Yang-Baxter operator. From now on, we assume  $k$  is a commutative ring with identity whenever we deal with Yang-Baxter operators.

**Definition 1.2.** Let  $X$  be a set. If  $R : X \times X \rightarrow X \times X$  is a function that satisfies  $(R \times Id_X) \circ (Id_X \times R) \circ (R \times Id_X) = (Id_X \times R) \circ (R \times Id_X) \circ (Id_X \times R)$ , then we say  $R$  is a pre-set-theoretic Yang-Baxter operator and the equation above is a set-theoretic Yang-Baxter equation. If, in addition,  $R$  is invertible, then we say  $R$  is a set-theoretic Yang-Baxter operator.

Pre-set-theoretic Yang-Baxter operator leads to Yang-Baxter operator by putting  $V = kX$ , and extending  $R : X \times X \rightarrow X \times X$  to  $R : V \otimes V \rightarrow V \otimes V$ . This Yang-Baxter operator is still called pre-set-theoretic Yang-Baxter operator.

## 2. SET-THEORETIC YANG-BAXTER HOMOLOGY THEORIES

Given a pre-set-theoretic Yang-Baxter operator  $R$ , we have two approaches to homology built on  $R$ . The “algebraic” version defined in [CES] and the “Graphic” version in [Prz-2]. We discuss them in the next two subsections. We prove the equivalence of them in Section 3.

We first review the “algebraic” version of set-theoretic Yang-Baxter homology theory based on [CES] and then introduce the “graphic” version of set-theoretic Yang-Baxter homology theory.

### 2.1. “Algebraic” homology of Carter, Elhamdadi, and Saito.

**Definition 2.1.** The set  $X$  together with a set-theoretic Yang-Baxter operator  $R$ ,  $(X, R)$ , is called in [CES] a Yang-Baxter set. We represent a function  $R$  by  $R(x_1, x_2) = (R_1(x_1, x_2), R_2(x_1, x_2))$ .

We use the following notation. Let  $\mathcal{I}_n$  be the  $n$ -dimensional cube  $I^n$  ( $I = [0, 1]$ ) regarded as a CW (cubical) complex, where  $n$  is a positive integer.<sup>1</sup> Denote the  $k$ -skeleton by  $\mathcal{I}_n^{(k)}$  with orientation given by the order of coordinate axes. In particular, every 2-face can be written as

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<sup>1</sup>We deal here with a co-pre-cubic set  $(X_k, d_\epsilon^i)$  where  $X_k = I^k$  and co-face maps  $d_\epsilon^{i,k-1} : X_{k-1} \rightarrow X_k$  defined by  $d_\epsilon^i(x_1, x_2, \dots, x_{k-1}) = (x_1, x_2, \dots, x_{i-1}, \epsilon, x_i, \dots, x_{k-1})$ ; they satisfy  $d_\delta^j d_\epsilon^i = d_\epsilon^i d_\delta^{j-1}$  where  $i < j$ .

$\epsilon_1 \times \cdots \times \epsilon_{i-1} \times I_i \times \epsilon_{i+1} \times \cdots \times \epsilon_{j-1} \times I_j \times \epsilon_{j+1} \times \cdots \times \epsilon_n$ , for some  $1 \leq i < j \leq n$ , where  $\epsilon_k = 0$  or  $1$ , and  $I_i, I_j$  denote two copies of  $I$  at the  $i$ th,  $j$ th positions, respectively.

**Definition 2.2.** The Yang-Baxter coloring of  $\mathcal{I}_n$  by a Yang-Baxter set  $(X, R)$  is a map  $L : E(\mathcal{I}_n) \rightarrow X$ , where  $E(\mathcal{I}_n)$  denotes the set of edges of  $\mathcal{I}_n$ , with each edge oriented as above, such that if

$$L(\epsilon_1 \times \cdots \times \epsilon_{i-1} \times I_i \times \epsilon_{i+1} \times \cdots \times \epsilon_n) = x,$$

$$L(\epsilon_1 \times \cdots \times \epsilon_{j-1} \times I_j \times \epsilon_{j+1} \times \cdots \times \epsilon_n) = y,$$

then

$$L(\epsilon_1 \times \cdots \times \epsilon_{i-1} \times 0_i \times \epsilon_{i+1} \times \cdots \times \epsilon_n) = R_1(x, y),$$

$$L(\epsilon_1 \times \cdots \times \epsilon_{j-1} \times 1_j \times \epsilon_{j+1} \times \cdots \times \epsilon_n) = R_2(x, y),$$

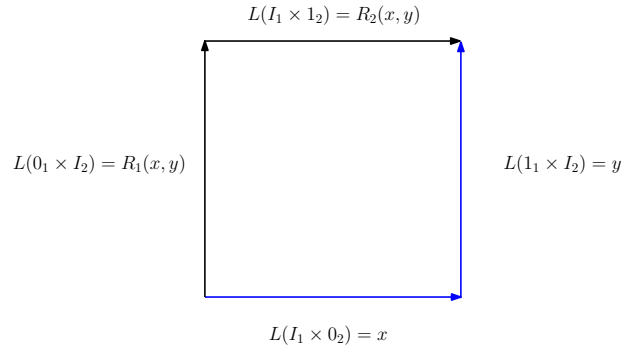


Figure. 1.1; local behavior of Yang-Baxter coloring

**Definition 2.3.** The initial path in  $\mathcal{I}_n$  is the sequence of edges of  $\mathcal{I}_n, (e_1, \dots, e_n)$ , where

$$e_1 = I_1 \times 0_2 \times \cdots \times 0_n,$$

$$e_2 = 1_1 \times I_2 \times 0_3 \times \cdots \times 0_n,$$

$\vdots$

$$e_n = 1_1 \times 1_2 \times \cdots \times 1_{n-1} \times I_n,$$

**Lemma 2.4.** (*S. Carter, M. Elhamdadi, and M. Saito, 2004*):

Let  $(X, R)$  be a Yang-Baxter set, and  $(e_1, \dots, e_n)$  be the initial path of  $\mathcal{I}_n$ . For any  $n$ -tuple of elements of  $X, (x_1, \dots, x_n)$ , there exists a unique Yang-Baxter coloring  $L$  of  $\mathcal{I}_n$  by  $(X, R)$  such that  $L(e_i) = x_i$  for all  $i = 1, \dots, n$ .

This lemma gives the following two properties.

- (1) Each edge has the color uniquely induced by the  $n$ -tuple associated to the initial path of  $\mathcal{I}_n$ .

- (2) Each  $k$ -face  $\mathcal{J}$  of  $\mathcal{I}_n$  has its induced initial path determined by the order of coordinates. Therefore, we can associate to it the  $k$ -tuple  $(y_1, \dots, y_k)$  determined by colors on its induced initial path. Denote this situation by  $L(\mathcal{J}) = (y_1, \dots, y_k)$

From these two facts, we have a way to map an  $n$ -tuple to  $(n-1)$ -tuple through the face maps in cubic homology theory.

Recall that  $\partial_n^C$  denotes the  $n$ -dimensional boundary map in the cubical homology theory. Thus  $\partial_n^C(\mathcal{I}_n) = \sum_{i=1}^{2n} \epsilon_i \mathcal{J}_i$ , where  $\mathcal{J}_i$  is an  $(n-1)$ -face and  $\epsilon_i = \pm 1$  depending on whether the orientation of  $\mathcal{J}_i$  matches the induced orientation. For the induced orientation, we take the convention that the inward pointing normal to an  $(n-1)$ -face appears last in a sequence of vectors that specifies an orientation, and the orientation of the  $(n-1)$ -face is chosen so that this sequence agrees with the orientation of the  $n$ -cube.

Let  $(X, R)$  be a Yang-Baxter set. Let  $C_n^{YB}(X)$  be the free abelian group generated by  $n$ -tuples  $(x_1, \dots, x_n)$  of elements of  $X$ . Define a homomorphism  $\partial_n^A : C_n^{YB}(X) \rightarrow C_{n-1}^{YB}(X)$  by  $\partial_n^A((x_1, \dots, x_n)) = L(\partial_n^C(\mathcal{I}_n)) = \sum_{i=1}^{2n} \epsilon_i L(\mathcal{J}_i)$ . We have  $\partial_{n-1}^A \circ \partial_n^A = 0$ , and  $(C_n^{YB}(X), \partial_n^A)$  is a chain complex. As usual, we can define  $H_n^A = \ker \partial_n^A / \text{im} \partial_{n+1}^A$  to be the “algebraic” version of Yang-Baxter homology group [CES].

**2.2. “Graphic” approach to Yang-Baxter homology.** In this homology theory, the chain groups are the same as before, that is  $C_n^{YB}(X) = ZX^n$ . We define the boundary homomorphism  $\partial_n^G : C_n^{YB}(X) \rightarrow C_{n-1}^{YB}(X)$  as follow,  $\partial_n^G = \sum_{i=1}^n (-1)^i d_{i,n}$ , where  $d_{i,n} = d_{i,n}^l - d_{i,n}^r$ . We can interpret the face maps through Figure 2.1

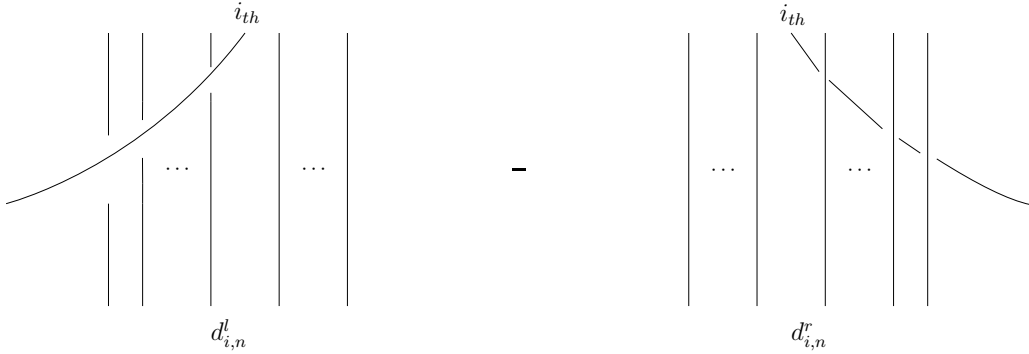
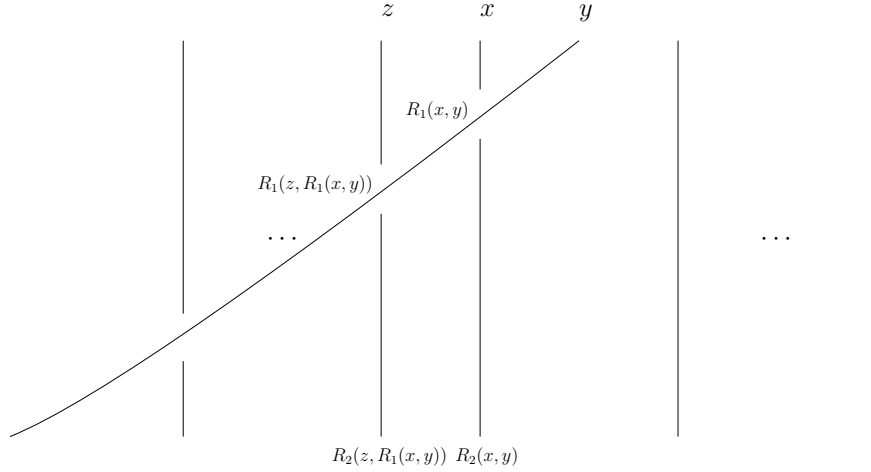
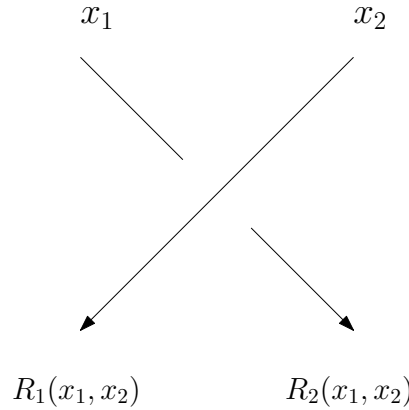


Figure. 2.1; a face map  $d_{i,n}$

The meaning of  $d_{i,n}^l$ , is illustrated in Figure 2.2;  $d_{i,n}^r$  can be described similarly.


 Figure. 2.2; a face map  $d_{i,n}^l$ 

We have an  $n$ -tuple as an input and each strand carries the corresponding element of the  $n$ -tuple. We track down the graph from top to bottom, and at each crossing we apply the fixed Yang-Baxter operator with input the ordered pair consists of two elements carried by the two strands right above the crossing (as in Figure 2.3).


 Figure. 2.3; encoding  $R$  at each crossing

Then the left strand after the crossing carries the  $R_1$  function value and the right strand after the crossing carries the  $R_2$  function value. In the end, we ignore the element carried by the left most strand, and this procedure generates an  $(n - 1)$ -tuple consisting of the  $n - 1$  elements carried by the other  $n - 1$  strands at the bottom.

One can easily check  $(X^n, d_{i,n}^\epsilon)$  form a pre-cubic set, which implies that  $(C_*^{YB}(X), \partial_n^G)$  is a chain complex. We define  $H_n^G = \ker \partial_n^G / \text{im} \partial_{n+1}^G$  as the “graphic” version of Yang-Baxter homology group.

## 3. EQUIVALENCE OF TWO HOMOLOGY THEORIES

**Theorem 3.1.** *The “algebraic” and “graphic” definitions of Yang-Baxter homology coincide. More generally, chain complexes leading to above homologies are isomorphic.*

*Proof.* Consider an  $n$ -dimensional cube  $I^n$  and denote it as  $I_1 \times \cdots \times I_n$ . For any coloring, say  $L(I_1 \times \cdots \times I_n) = (x_1, \dots, x_n)$ , and an  $(n-1)$ -face  $\mathcal{J} = I_1 \times \cdots \times I_{i-1} \times 1_i \times I_{i+1} \times \cdots \times I_n$ , we are going to demonstrate that  $L(\mathcal{J}) = d_{i,n}^l(x_1, \dots, x_n)$ . To see this, we need to calculate the coloring of the initial path  $(a_1, \dots, a_{n-1})$  of this  $(n-1)$ -face. By definition,

$$\begin{aligned} a_1 &= I_1 \times 0_2 \times 0_3 \times \cdots \times 0_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n \\ a_2 &= 1_1 \times I_2 \times 0_3 \times \cdots \times 0_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n \\ &\vdots \\ a_{i-1} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times I_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n \\ a_i &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times I_{i+1} \times 0_{i+2} \times \cdots \times 0_n \\ a_{i+1} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times 1_{i+1} \times I_{i+2} \times \cdots \times 0_n \\ &\vdots \\ a_{n-2} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times 1_{i+1} \times \cdots \times I_{n-1} \times 0_n \\ a_{n-1} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-1} \times 1_i \times 1_{i+1} \times \cdots \times 1_{n-1} \times I_n \end{aligned}$$

We need another sequence  $(b_2, \dots, b_i)$ , where

$$\begin{aligned} b_2 &= 1_1 \times 0_2 \times 0_3 \times \cdots \times 0_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\ b_3 &= 1_1 \times 1_2 \times 0_3 \times \cdots \times 0_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\ &\vdots \\ b_{i-1} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\ b_i &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 1_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \end{aligned}$$

For example, for  $j = i$ , the edges of the square are

$$\begin{aligned} e_{i-1} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times I_{i-1} \times 0_i \times 0_{i+1} \times \cdots \times 0_n \\ b_i = e_i &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 1_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\ b_{i-1} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times 1_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\ a_{i-1} &= 1_1 \times 1_2 \times 1_3 \times \cdots \times I_{i-1} \times 1_i \times 0_{i+1} \times \cdots \times 0_n \end{aligned}$$

Since

$$L(e_{i-1}) = x_{i-1}, \quad L(b_i) = x_i$$

we have

$$\begin{aligned} L(a_{i-1}) &= R_2(L(e_{i-1}), L(b_i)) = R_2(x_{i-1}, x_i) \\ L(b_{i-1}) &= R_1(L(e_{i-1}), L(b_i)) = R_1(x_{i-1}, x_i) \end{aligned}$$

Once we know the color of  $b_j$ , we know the colors of  $b_{j-1}$ , and  $a_{j-1}$  (at each iteration we deal with a square in Figure 3.1). Thus, recursively, we know all the colors of  $a_j$ 's i.e. we get an  $(n-1)$ -tuple.

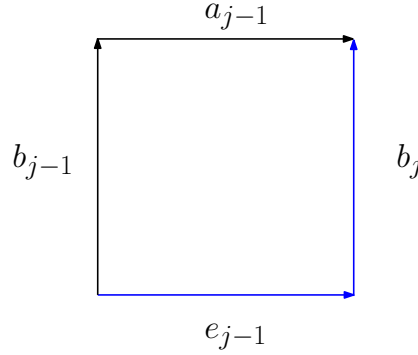


Figure. 3.1

In general, we have

$$\begin{aligned} L(a_j) &= \begin{cases} R_2(L(e_j), L(b_{j+1})) & 1 \leq j \leq i-1 \\ x_{j+1} & i \leq j \leq n-1 \end{cases} \\ L(b_j) &= \begin{cases} R_1(L(e_j), L(b_{j+1})) & 2 \leq j \leq i-1 \\ x_j & j = i \end{cases} \end{aligned}$$

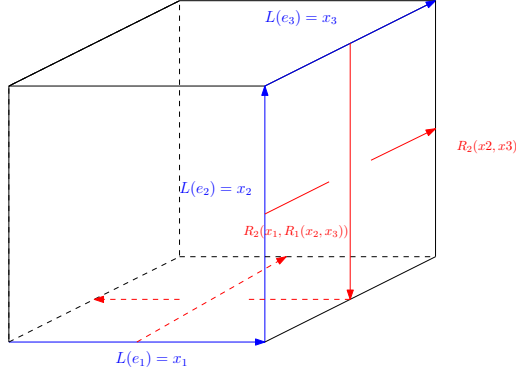
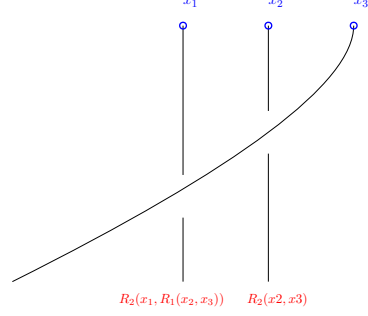
We can see that this  $(n-1)$ -tuple is the same as the one given by  $d_{i,n}^l$  (compare with Figure 2.2).

Similarly, if we consider  $\mathcal{J} = I_1 \times \cdots \times I_{i-1} \times 0_i \times I_{i+1} \times \cdots \times I_n$ , we can show  $L(\mathcal{J}) = d_{i,n}^r(x_1, \dots, x_n)$ .

As for the sign, we can directly calculate that, for  $L(I_1 \times \cdots \times I_{i-1} \times \epsilon_i \times I_{i+1} \times \cdots \times I_n)$ , the sign is  $(-1)^{n-i}(1-2\epsilon_i)$ .

Thus, the boundary map of “algebraic” version and the boundary map of “graphic” version only differ by a global sign, therefore, the considered chain complexes are isomorphic and they give isomorphic homology groups.  $\square$

**Example 3.2.** Comparison of the face maps corresponding to the 2-face  $I_1 \times I_2 \times 1_3$  of a cube and  $d_{3,3}^l$ .

Figure. 3.2; coloring of face  $I \times I \times 1$ Figure. 3.3; face map  $d_{3,3}^l$ 

From the figure above, we can see a “curtain-like” object similar to the figure on the right hand side “climbing” on the faces of the cube. For higher dimensions, the similar consideration holds. Therefore this also gives a way to visualize the equivalence.

#### 4. HOMOLOGY FOR UNITAL YANG-BAXTER OPERATORS

This “graphic” definition of Yang-Baxter homology was motivated by the homology theory of self-distributive systems [Prz-1, Prz-2], for example shelves, racks, and quandles. Similarly, we can define one-term and two-term homology not only for set-theoretic Yang-Baxter operators but also for pre-Yang-Baxter operators. Furthermore, we can extend the definition of pre-Yang-Baxter homology to pre-Yang-Baxter operators with Yang-Baxter wall (see Definition 4.2). We will give the general definitions below and discuss some properties of these homology theories.

**4.1. One-term Yang-Baxter homology.** Let  $k$  be a commutative ring with identity,  $V = kX$  be a free  $k$  module with basis  $X$  and  $M$  be a right  $k$ -module. We define in Definition 4.2 the one-term pre-Yang-Baxter chain complex  $\mathcal{C}^{YB} = (C_n, M, \partial_n)$  from the pre-simplicial module  $(C_n, M, d_i)$ .

First we recall the notion of a pre-simplicial module.

**Definition 4.1.** The pre-simplicial module  $\mathcal{M}$  is a collection of modules  $M_n$ ,  $n \geq 0$ , together with maps, called face maps or face operators,

$$d_i : M_n \rightarrow M_{n-1}, \quad 0 \leq i \leq n,$$

such that:

$$d_i d_j = d_{j-1} d_i, \quad 0 \leq i < j \leq n,$$



we define a chain complex with chain modules  $M_n$  and a boundary map  $\partial_n : M_n \rightarrow M_{n-1}$  given by:

$$\partial_n = \sum_{i=0}^n (-1)^i d_i$$

We are ready to define the pre-Yang-Baxter pre-simplicial module.

**Definition 4.2.** ([Leb, Prz-2]) Consider a linear map  $R^W : M \otimes V \rightarrow M$ , such that  $R^W \circ (R^W \otimes id_V) = R^W \circ (R^W \otimes id_V) \circ (id_M \otimes R)$  as shown graphically in Figure 4.1, we call this the left wall condition.

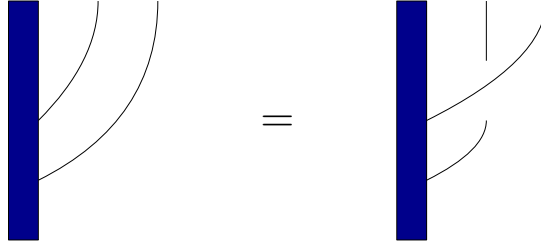


Figure. 4.1; the left wall condition

Let  $C_n = M \otimes V^{\otimes n}$  and the face map  $d_i = d_{i,n} : C_n \rightarrow C_{n-1}$  is defined by

$$d_i = (R^W \otimes id^{\otimes n-1}) \circ (id_M \otimes R \otimes id^{\otimes n-2}) \circ (id_M \otimes id \otimes R \otimes id^{\otimes n-3}) \circ \dots \circ (id_M \otimes id^{\otimes i-2} \otimes R \otimes id^{n-i})$$

We can interpret the face maps through Figure 4.2

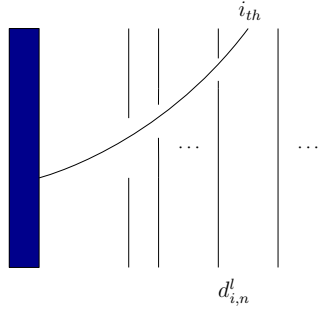


Figure. 4.2; face map  $d_{i,n}$

$(C_n, M, d_i)$  is a pre-simplicial module, and  $(C_n, M, \partial_n)$  is the one-term pre-Yang-Baxter chain complex. It's homology is called the one-term pre-Yang-Baxter homology  $H_n(R, R^W)$ .

**Proposition 4.3.** Let  $(C_n, M, \partial_n)$  be the chain complex of the one-term pre-Yang-Baxter homology. For a fixed element  $v \in V$ , consider

the map  $f_n : C_n \rightarrow C_n$ , defined by  $f_n(a) = d_{n+1,n+1}(a \otimes v)$ , where  $a \in C_n$ , then this is a chain map. We have

- (1)  $(f_n)_*(H_n) = 0$ .
- (2) If there is an element  $v \in V$  such that  $f_n$  is invertible, then the one-term pre-Yang-Baxter homology is trivial.

*Proof.* We construct a chain homotopy  $P_n$  between  $(-1)^{n+1}f_n$  and zero map, in particular showing that  $f_n$  is a chain map. The chain homotopy  $P_n = C_n \rightarrow C_{n+1}$  is defined by  $P_n(v_{i_1} \otimes \dots \otimes v_{i_n}) = (-1)^n v_{i_1} \otimes \dots \otimes v_{i_n} \otimes v$ . We check:  $\partial_{n+1}P_n + P_{n-1}\partial_n = \sum_{i=1}^{n+1} (-1)^i d_{i,n+1}P_n + \sum_{j=1}^n (-1)^j P_n d_{j,n} = (-1)^{n+1} d_{n+1,n+1}P_n = (-1)^{n+1}f_n$ . (1) follows because  $\{f_n\}$  is chain homotopic to the 0 map, therefore,  $(f_n)_*$  is the 0 map on homology. (2) follows since if  $f_n$  is invertible, so is  $(-1)^{n+1}f_n$ , thus  $H_n(C_*)$  is isomorphic to  $H_n(C_*)$  through zero map. This shows that  $H_n(C_*) = 0$ .  $\square$

In the case that  $M = k$ , and of  $V$  acting on  $k$  trivially, the condition making  $(V^{\otimes n}, d_{i,n})$  a pre-simplicial module is equivalent to that the sum of each column of the  $R$  matrix is 1, which we call the column unital condition (e.g. stochastic matrices satisfy the condition).

**Corollary 4.4.** *Let  $M = k$ ,  $V$  act on  $k$  trivially, and  $R$  be a set-theoretic Yang-Baxter operator. If for any pair  $(B, D)$ , there is a unique  $A$  such that  $R(A, B) = (R_1(A, B), R_2(A, B)) = (C, D)$ , then conditions in Proposition 4.3 hold. In particular, biracks satisfy this condition (see Definition 3.1 condition 3, that is right invertibility, in [CES]).*

*Proof.* We need to show that for any  $n$ -tuple  $(y_1, \dots, y_n)$  in  $C_n$ , there exists unique  $n$ -tuple  $(x_1, \dots, x_n)$  in  $C_n$  such that  $f_n((x_1, \dots, x_n)) = d_{n+1,n+1}((x_1, \dots, x_n, v)) = (y_1, \dots, y_n)$ . Since we know  $y_n$  and  $v$ , we get the values of  $x_n$  and  $R_1(x_n, v)$  uniquely. Once we have  $R_1(x_n, v)$ , together the value of  $y_{n-1}$ , we get the value of  $x_{n-1}$  and  $R_1(x_{n-1}, R_1(x_n, v))$  uniquely. Thus by this iteration, we get the  $n$ -tuple  $(x_1, \dots, x_n)$  uniquely and this shows  $f_n$  is invertible.  $\square$

**Example 4.5.** (Compare [CES]) Let  $F$  be a commutative ring with identity. Let  $k = F[s^{\pm 1}, t^{\pm 1}]/(1-s)(1-t)$ , then

$$R(x, y) = (R_1(x, y), R_2(x, y)) = ((1-s)x + ty, sx + (1-t)y)$$

is a set-theoretic Yang-Baxter operator satisfying the conditions in Corollary 4.4. This holds because for any given  $y$  and  $a = R_2(x, y) = sx + (1-t)y$ , we can solve  $x = s^{-1}(a - (1-t)y)$ . Thus the one-term homology of this operator is trivial.

*Remark 4.6.* Pre-Yang-Baxter coming from racks  $(X, *)$  where  $R(a, b) = (b, a * b)$  satisfies the conditions in Proposition 4.3. Thus it has zero one-term homology (see [Prz-1]).

**4.2. Two-term Yang-Baxter homology.** Let  $k$  be a commutative ring with identity,  $V = kX$  be a free  $k$  module with basis  $X$ ,  $M$  be a right  $k$ -module and  $N$  be a left  $k$ -module. We define in Definition 4.8 the two-term pre-Yang-Baxter chain complex  $\mathcal{C}^{YB} = (C_n, M, N, \partial_n)$  from the pre-cubical module  $(C_n, M, N, d_i^\epsilon)$ .

**Definition 4.7.** The pre-cubical module  $\mathcal{M}$  is a collection of modules  $M_n$ ,  $n \geq 0$ , together with maps, called face maps or face operators,

$$d_i^\epsilon : M_n \rightarrow M_{n-1}, \quad 1 \leq i \leq n, \epsilon = 0, 1$$

such that:

$$d_i^\epsilon d_j^\delta = d_{j-1}^\delta d_i^\epsilon, \quad 1 \leq i < j \leq n, \epsilon, \delta = 0, 1$$

we define a chain complex with chain groups  $M_n$  and a boundary map  $\partial_n : M_n \rightarrow M_{n-1}$  given by:

$$\partial_n = \sum_{i=1}^n (-1)^i (d_i^0 - d_i^1).$$

We are ready to define the pre-Yang-Baxter pre-cubical module.

**Definition 4.8.** ([Leb, Prz-2]) Consider a linear map  $R_l^W : M \otimes V \rightarrow M$ , such that  $R_l^W \circ (R_l^W \otimes id_V) = R_l^W \circ (R_l^W \otimes id_V) \circ (id_M \otimes R)$ , and a linear map  $R_r^W : V \otimes N \rightarrow N$ , such that  $R_r^W \circ (id_V \otimes R_r^W) = R_r^W \circ (id_V \otimes R_r^W) \circ (R \otimes id_N)$  we call them the left and right wall conditions (graphically see Figure 4.3 and Figure 4.4).

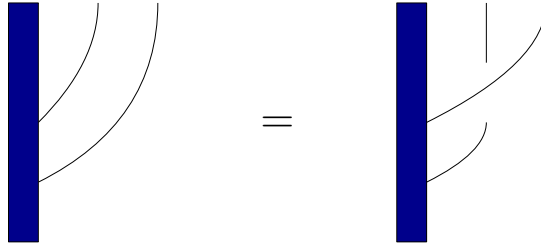


Figure. 4.3; left wall condition

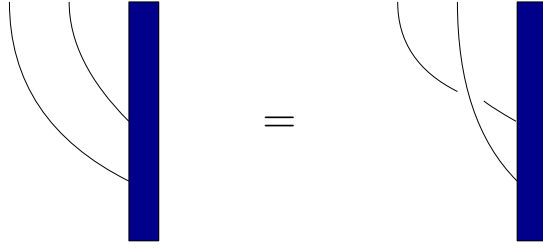


Figure. 4.4; right wall condition

Let  $C_n = M \otimes V^{\otimes n} \otimes N$  and face maps  $d_i^\epsilon : C_n \rightarrow C_{n-1}$  are given by

$$d_i^l = (R_l^W \otimes id^{\otimes n-2} \otimes id_N) \circ (id_M \otimes R \otimes id^{\otimes n-1} \otimes id_N) \circ (id_M \otimes id \otimes R \otimes id^{\otimes n-3} \otimes id_N) \circ \dots$$

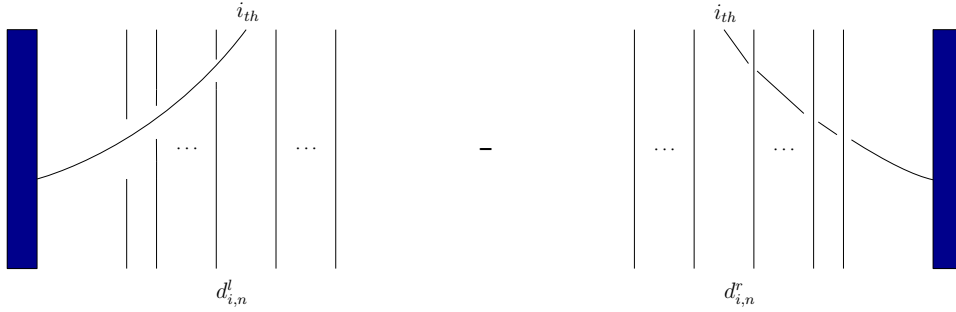
$$\dots \circ (id_M \otimes id^{\otimes i-2} \otimes R \otimes id^{\otimes n-i} \otimes id_N)$$

and

$$d_i^r = (id_M \otimes id^{\otimes n-1} \otimes R_r^W) \circ (id_M \otimes id^{\otimes n-2} \otimes R \otimes id_N) \circ (id_M \otimes id^{\otimes n-3} \otimes R \otimes id \otimes id_N) \circ \dots$$

$$\dots \circ (id_M \otimes id^{\otimes i-1} \otimes R \otimes id^{\otimes n-i-1} \otimes id_N)$$

We can interpret the face maps  $d_i^l$ ,  $d_i^r$  and their difference through Figure 4.5.

Figure. 4.5; face map  $d_{i,n+1}$ 

$(C_n, M, N, d_i^\epsilon)$  is a pre-cubical module, and  $(C_n, M, N, \partial_n)$  is the two-term pre-Yang-Baxter chain complex.

In the case that  $M = k = N$ , and the action of  $V$  on  $k$  is the trivial action, the conditions making  $(V^{\otimes n}, d_{i,n}^\epsilon)$  a pre-cubical module is equivalent to saying that  $R$  has the column unital condition.

**Example 4.9.** We give a family of unital Yang-Baxter operator  $R : V \otimes V \rightarrow V \otimes V$ , where  $V = k \{v_1, \dots, v_m\}$ ,  $k = \mathbb{Q}[y, y^{-1}]$  and  $m$

is a positive integer. For any given  $m$ ,  $R$  can be represented by its coefficients,

$$R_{ij}^{kl} = \begin{cases} 1, & \text{if } i=j=k=l; \\ 1, & \text{if } l=i>j=k; \\ y^2, & \text{if } l=i<j=k; \\ 1-y^2, & \text{if } k=i<j=l; \\ 0, & \text{otherwise.} \end{cases}$$

These family of Yang-Baxter operators are unital, for example when  $m = 2$ , it is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-y^2 & 1 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Computation shows interesting pattern in the two-term Yang-Baxter homology of this family of Yang-Baxter operators.

**Conjecture 4.10.** ([Wan]) *When  $m = 2$ ,  $H_n = k^2 \oplus (k/(1-y^2))^{a_n} \oplus (k/(1-y^4))^{s_{n-2}}$ , where  $s_n = \sum_{i=1}^{n+1} f_i$  is the partial sum of Fibonacci sequence, where  $f_1 = 1 = f_2$  and  $a_n$  is given by  $2^n = 2 + a_{n-1} + s_{n-2} + a_n + s_{n-1}$  with  $a_1 = 0$ . We verified the conjecture for  $n \leq 10$ .*

What fascinates us is that this family of Yang-Baxter operators come from the Yang-Baxter operators giving  $sl_m$  polynomial invariants of links (substitutions to the Homflypt polynomial). For example, when  $m = 2$ , the matrix is

$$\begin{bmatrix} -q & 0 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q \end{bmatrix}$$

If we divide elements in each column by the sum of those elements and make the substitution  $y = (1 + q^{-1} - q)^{-1/2}$ , we will get the matrix in Example 4.9. In general, we can get our family in Example 4.9 in a similar way (normalizing columns) and again they are Yang-Baxter operators. More interestingly, this new family of Yang-Baxter operators also provide  $sl_m$  polynomial invariants of links [Wan]. This fact is implicit in [Jon-2].

**4.3. Cocycle invariant from Yang-Baxter homology.** In the paper [CES] where Carter, Elhamdadi and Saito define their set-theoretic Yang-Baxter homology, they also define a 2-cocycle link invariant from the set-theoretic Yang-Baxter homology. This can also be done in the

case of column unital Yang-Baxter operators. In [Prz-3], it demonstrates the third Reidemeister move preserve the (co)homology of the (co)cycle constructed from a knot diagram. See Figure 4.6 and [Prz-3] for details.

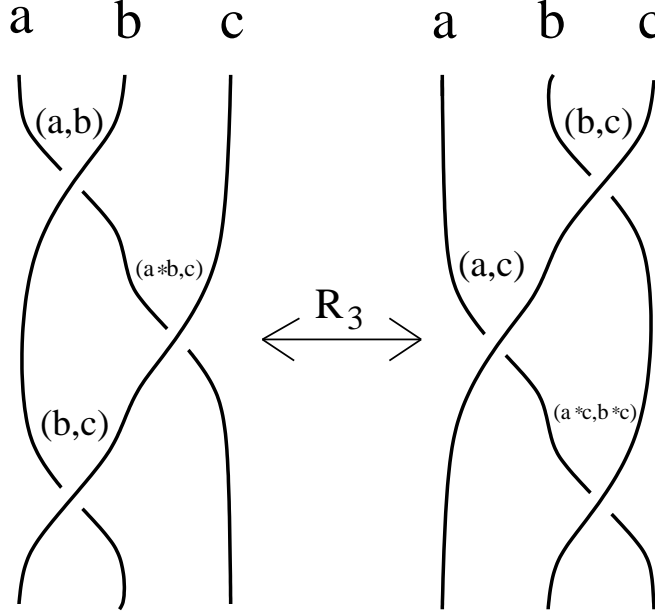


Figure. 4.6; relation between third Reidemeister move and 2-(co)cycle condition

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DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY,  
WASHINGTON DC, USA AND UNIVERSITY OF GDAŃSK  
*E-mail address:* przytyck@gwu.edu

DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY,  
WASHINGTON DC, USA  
*E-mail address:* wangxiao@gwu.edu